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Some Lower Bounds for the Complexity of Continuation Methods

Jean-Pierre Dedieu*

LAO, Université Paul Sabatier, 31062 Toulouse Cedex 04, France
E-mail: dedieu@cict.fr

and

Steve Smale†

Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Hong Kong
E-mail: masmale@math.cityu.edu.hk

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1. INTRODUCTION AND MAIN RESULTS

In this note we consider the zero-finding problem for a homogeneous polynomial system,

$$f\colon \mathbb{C}^{n+1} \to \mathbb{C}^m,$$

with $m \leqslant n$, $f = (f_1, ..., f_m)$, $f_i \in \mathscr{H}_{d_i}$: the space of homogeneous polynomials $f_i\colon \mathbb{C}^{n+1} \to \mathbb{C}$ with degree$(f_i) = d_i$. The well-determined $(m = n)$ and under-determined $(m < n)$ cases are considered together. We also let $D = \max d_i$, $d = (d_1, ..., d_m)$, and $\mathscr{H}_d = \mathscr{H}_{d_1} \times \cdots \times \mathscr{H}_{d_m}$.

The projective Newton method has been introduced by Shub in [6] and is defined by

$$N_f(x) = x - (Df(x)|_{x^\perp})^{-1}\, f(x)$$

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† Corresponding author.

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when $m=n$, with $Df(x)|_{x^\perp}$ the restriction at $x^\perp$ of the derivative of f at x. Here $x^\perp$ denotes the space orthogonal to x in $\mathbb{C}^{n+1}$. When $m \leqslant n$, we take

$$N_f(x) = x - (Df(x)|_{x^\perp})^\dagger f(x),$$

where, for any linear operator $A\colon \mathbb{E} \to \mathbb{F}$ between two Hermitian spaces, $A^\dagger$ denotes its Moore–Penrose inverse. $A^\dagger = A^\star (AA^\star)^{-1}$ when A is onto, which is the case considered here.

A Newton continuation method sequence (NCM sequence) is a sequence of pairs

$$(f_i, \zeta_i) \in (\mathcal{H}_d)^\star \times (\mathbb{C}^{n+1})^\star, \qquad 0 \leqslant i \leqslant k$$

(given a vector space E, $E^\star$ denotes the set of nonzero vectors) satisfying the conditions

$$f_i(\zeta_i) = 0, \qquad 0 \leqslant i \leqslant k$$

and

$$\alpha(f_{i+1}, \zeta_i) \leqslant \alpha_0, \qquad \text{with associated zero} \quad \zeta_{i+1}, \qquad 0 \leqslant i \leqslant k-1.$$

This last condition, we will make it precise later, implies that the projective Newton's sequence

$$x_0 = \zeta_i, \qquad x_{p+1} = N_{f_{i+1}}(x_p), \qquad p \geqslant 0,$$

converges quadratically towards ζ_{i+1}.

The complexity of an NCM sequence (f_i, ζ_i), $0 \leqslant i \leqslant k$, is measured by k. Upper bounds for the complexity of NCM sequences have been given by Shub and Smale in their papers [7–9], about the complexity of Bézout's theorem. They give an upper bound, depending mainly on the degree D of the considered system and on the condition number of the homotopy. The case of sparse polynomial systems is studied in [2] by Dedieu; the case of homogeneous polynomial systems by Malajovich [5] and Blum, Cucker, Shub, and Smale in their book [1, Chap. 14], the case of multi-homogeneous underdetermined polynomial systems by Dedieu and Shub [3] and the case of overdetermined polynomial systems by Dedieu and Shub [4].

Our main results here are two lower bounds for the complexity of a NCM sequence. In the first one we relate this complexity to the degree:

THEOREM 1. *For any NCM sequence* (f_i, ζ_i), $0 \leqslant i \leqslant k$, *one has*

$$k \geqslant c \max\left(1, \frac{D-1}{2}\right) d_R(\zeta_0, \zeta_k),$$

where $c > 0$ *is a universal constant given below and* $d_R(\zeta_0, \zeta_k)$ *the Riemannian distance in* $\mathbb{P}(\mathbb{C}^{n+1})$ *between* ζ_0 *and* ζ_k.

The Riemannian distance in $\mathbb{P}(\mathbb{C}^{n+1})$ is defined by

$$d_R(u, v) = \arccos \frac{|\langle u, v \rangle|}{\|u\| \, \|v\|}$$

for any $u, v \in (\mathbb{C}^{n+1})^\star$.

Remark 1. A direct computation shows that this bound is sharp and this complexity is obtained for the family of systems defined by

$$f_{t,j}(z_0, z_1, ..., z_n) = z_0^{d_j - 1}(z_j - \zeta_{t,j} z_0), \qquad 1 \leqslant j \leqslant n,$$

where $\zeta_t = (1-t)(1, 0, ..., 0) + t(1, a_1, ..., a_n)$, $a \in \mathbb{C}^n$ given.

For any $f \in \mathcal{H}_d$ let us define

$$\Sigma_f = \{x \in (\mathbb{C}^{n+1})^\star : \text{rank } Df(x) < m\}.$$

In our second theorem, we give a lower bound for the complexity of an NCM sequence in terms of the arithmetic mean of the distances of ζ_i from Σ_{f_i}.

THEOREM 2. *For any NCM sequence* (f_i, ζ_i), $0 \leqslant i \leqslant k$, *one has*

$$k \geqslant c \frac{d_R(\zeta_0, \zeta_k)}{k^{-1} \sum_{i=1}^{k} d_R(\zeta_i, \Sigma_{f_i})},$$

where $c > 0$ *is (another) universal constant.*

Remark 2. This lower bound shows that the complexity of an NCM sequence increases with the proximity of singular points. This proximity is measured here by the arithmetic mean of the distances $d_R(\zeta_i, \Sigma_{f_i})$, $1 \leqslant i \leqslant k$.

COROLLARY 1. *Let* $\varepsilon > 0$ *be given. For any NCM sequence* (f_i, ζ_i), $0 \leqslant i \leqslant k$, *such that*

$$d_R(\zeta_i, \Sigma_{f_i}) \leqslant \varepsilon, \qquad 1 \leqslant i \leqslant k,$$

we have

$$k \geqslant c\varepsilon^{-1}\, d_R(\zeta_0, \zeta_k),$$

with c as in Theorem 2.

The proofs of these theorems are based on Smale's alpha-theory introduced by Smale in [10]. We use here its homogeneous version as described in Dedieu and Shub [3]. When $Df(x)$ is onto, we define

$$\gamma(f, x) = \max\left(1,\ \|x\| \max_{k \geqslant 2} \left\| (Df(x)|_{x^\perp})^\dagger \frac{D^k f(x)}{k!} \right\|^{1/(k-1)}\right),$$

$$\beta(f, x) = \|x\|^{-1}\ \|(Df(x)|_{x^\perp})^\dagger\, f(x)\|,$$

$$\alpha(f, x) = \beta(f, x)\, \gamma(f, x).$$

In the definition of $\gamma(f, x)$, $\|\ \|$ is the operator norm with respect to the canonical Hermitian structure over $\mathbb{C}^{n+1}$.

These three quantities are invariant under scaling and under unitary transformations,

$$\star(f, x) = \star(f, \lambda x) = \star(\lambda f, x) = \star(fou, u^{-1}(x)),$$

with $\star \in \{\alpha, \beta, \gamma\}$, for any $x \in (\mathbb{C}^{n+1})^\star$, $\lambda \in \mathbb{C}^\star$, and any unitary transformation u in $\mathbb{C}^{n+1}$. When $Df(x)$ is not onto, we take

$$\alpha(f, x) = \beta(f, x) = \gamma(f, x) = \infty.$$

The following theorem [3, Theorem 1] justifies our definition of a NPC sequence.

THEOREM 3. *There is a universal constant* $\alpha_0 > 0$ *with the following property: for any homogeneous system* $f \in \mathscr{H}_d$ *and* $x \in (\mathbb{C}^{n+1})^\star$, *if* $\alpha(f, x) \leqslant \alpha_0$, *then the projective Newton sequence*,

$$x_0 = x, \qquad x_{k+1} = N_f(x_k),$$

is defined and satisfies

$$\|x_{k+1} - x_k\| / \|x_k\| \leqslant (\tfrac{1}{2})^{2^k - 1}\, \beta(f, x)$$

for any $k \geqslant 0$. *This sequence converges to a zero* $\zeta \in (\mathbb{C}^{n+1})^\star$ *of f and*

$$d_R(\zeta, x_k) \leqslant \sigma(\tfrac{1}{2})^{2^k - 1}\, \beta(f, x)$$

with

$$\sigma = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{2^i - 1} = 1.6328....$$

We can take $\alpha_0 = 1/137$.

2. PROOFS OF THEOREMS 1 AND 2

These proofs are the consequences of the three following propositions. In the first one we compute the minimum value of $\gamma(f, \zeta)$.

PROPOSITION 1. *We have*

$$\max\left(1, \frac{D-1}{2}\right) = \min \gamma(f, \zeta),$$

where the minimum is taken over all pairs $(\zeta, f) \in (\mathbb{C}^{n+1})^{\star} \times (\mathscr{H}_d)^{\star}$ *with* $f(\zeta) = 0$.

Proof. We first prove that $(D-1)/2$ is a lower bound for $\gamma(f, \zeta)$. We can suppose that rank $Df(\zeta) = m$, since, otherwise, $\gamma(f, \zeta) = \infty$. Using the invariance properties of $\gamma(f, \zeta)$ under scaling and unitary transformations, we also can suppose that $\zeta = (1, 0, ..., 0)$. Since $f(\zeta) = 0$ we have

$$f_i(z) = z_0^{d_i - 1}(a_{i,1} z_1 + \cdots + a_{i,n} z_n) + g_i(z), \qquad 1 \leqslant i \leqslant m,$$

with degree$(g_i, z_0) \leqslant d_i - 2$. Let us denote by A the $m \times n$ matrix with entries $a_{i,j}$. Thus, $Df(\zeta) = (0 \mid A)$. The second derivative of $f = (f_1, ..., f_m)$ is given by

$$D^2 f_i(\zeta) = (d_i - 1) \begin{pmatrix} 0 & A_i \\ A_i^T & O_n \end{pmatrix} + D^2 g_i(\zeta),$$

where $A_i = (a_{i,1}, ..., a_{i,n})$ and O_n is the $n \times n$ zero matrix. Since degree$(g_i, z_0) \leqslant d_i - 2$ for any $v \in \mathbb{C}^{n+1}$, we have

$$D^2 f_i(\zeta)(\zeta, v) = (d_i - 1) \sum_{j=1}^{n} a_{i,j} v_j$$

so that

$$D^2 f(\zeta)(\zeta, v) = \mathrm{Diag}(d_i - 1)\, A\tilde{v}, \qquad \tilde{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Here $\mathrm{Diag}(d_i - 1)$ denotes the $m \times m$ diagonal matrix with diagonal enties $d_i - 1$. This gives

$$(Df(\zeta)|_{\zeta^\perp})^\dagger\, D^2 f(\zeta)(\zeta, v) = \begin{pmatrix} 0 \\ A^\dagger\, \mathrm{Diag}(d_i - 1)\, A\tilde{v} \end{pmatrix}$$

and, consequently,

$$\|A^\dagger\, \mathrm{Diag}(d_i - 1)\, A\|/2 = \|(Df(\zeta)|_{\zeta^\perp})^\dagger\, D^2 f(\zeta)/2\| \leqslant \gamma(f, \zeta).$$

Let us consider the matrix $B = A^\dagger\, \mathrm{Diag}(d_i - 1)\, A$. Let $A = U\Sigma V$ be a singular value decomposition of A: U and V are unitary $m \times m$ and $n \times n$ matrices and $\Sigma = (\Delta \mid 0)$ with $\Delta = \mathrm{Diag}(\sigma_i)$, $\sigma_1 \geqslant \cdots \geqslant \sigma_m > 0$ the singular values of A. We have

$$\begin{aligned} B &= V^\star \begin{pmatrix} \Delta^{-1} \\ 0 \end{pmatrix} U^\star\, \mathrm{Diag}(d_i - 1)\, U(\Delta \mid 0)\, V \\ &= V^\star \begin{pmatrix} \Delta^{-1} U^\star\, \mathrm{Diag}(d_i - 1)\, U\Delta & 0 \\ 0 & 0 \end{pmatrix} V, \end{aligned}$$

so that the eigenvalues of B are $d_1 - 1, ..., d_m - 1$ and 0. Since $\|B\| \geqslant \rho(B)$ (its spectral radius), we get

$$\|B\| = \|A^\dagger\, \mathrm{Diag}(d_i - 1)\, A\| \geqslant D - 1$$

and this proves the inequality

$$\gamma(f, \zeta) \geqslant \max\left(1, \frac{D - 1}{2}\right).$$

In order to prove the converse inequality, we study the example

$$f_i(z) = z_0^{d_i - 1} z_i, \qquad 1 \leqslant i \leqslant m, \qquad z = (z_0, ..., z_n),$$

and we consider $\zeta = (1, 0, ..., 0)$, so that $f(\zeta) = 0$. The derivative of f is given by

$$Df(\zeta) = (0, I_m, O_{n-m}),$$

with I_m the $m \times m$ identity matrix and O_{n-m} the $(n-m) \times (n-m)$ zero matrix. The other derivatives are given by

$$D^k f_i(\zeta)(u^1, ..., u^k) = \sum_{\substack{0 \leqslant i_j \leqslant n \\ 1 \leqslant j \leqslant k}} \frac{\partial^k f_i(\zeta)}{\partial z_{i_1} \cdots \partial z_{i_k}} u^1_{i_1} \cdots u^k_{i_k},$$

for any $u^1, ..., u^k \in \mathbb{C}^{n+1}$. These partial derivatives are equal to 0, except when, for some $j = 1 \cdots k$, we have

$$i_1 = \cdots = i_{j-1} = i_{j+1} = \cdots = i_k = 0, \qquad i_j = i.$$

In this case, this partial derivative is equal to $(d_i - 1) \cdots (d_i - k + 1)$. Thus,

$$D^k f_i(\zeta)(u^1, ..., u^k) = \sum_{j=1}^{k} (d_i - 1) \cdots (d_i - k + 1)\, u^1_0 \cdots u^{j-1}_0 u^j_i u^{j+1}_0 \cdots u^k_0$$

and

$$\begin{aligned} \|D^k f(\zeta)(u^1, ..., u^k)\| &= \left(\sum_{i=1}^{m} \left| \sum_{j=1}^{k} (d_i - 1) \cdots (d_i - k + 1) \right.\right. \\ &\qquad \left.\left. \times u^1_0 \cdots u^{j-1}_0 u^j_i u^{j+1}_0 \cdots u^k_0 \right|^2 \right)^{1/2} \\ &\leqslant (D-1) \cdots (D-k+1) \\ &\qquad \times \sum_{j=1}^{k} \left(\sum_{i=1}^{m} |u^1_0 \cdots u^{j-1}_0 u^j_i u^{j+1}_0 \cdots u^k_0|^2 \right)^{1/2} \\ &\leqslant (D-1) \cdots (D-k+1)\, k/2 \end{aligned}$$

when $\|u^1\| = \cdots = \|u^k\| = 1$, as can be proved by induction over $k \geqslant 2$. Consequently,

$$\|\zeta\| \max_{k \geqslant 2} \left\| (Df(\zeta)|_{\zeta^\perp})^\dagger \frac{D^k f(\zeta)}{k!} \right\|^{1/(k-1)} \leqslant \max_{k \geqslant 2} \left(\frac{1}{2} \binom{D-1}{k-1} \right)^{1/(k-1)} = \frac{D-1}{2},$$

using the fact that the sequence $(\frac{1}{2}\binom{D-1}{k-1})^{1/(k-1)}$, $k \geqslant 2$, is decreasing. This yields $\gamma(f, \zeta) \leqslant \max(1, (D-1)/2)$ and achieves the proof of Proposition 1. ∎

PROPOSITION 2. *There is a universal constant $c > 0$ with the following property: For any $f \in (\mathscr{H}_d)^\star$, ζ and $x \in (\mathbb{C}^{n+1})^\star$, if $\alpha(f, x) \leqslant \alpha_0$ with associated zero ζ, then*

$$d_R(\zeta, x)\, \gamma(f, \zeta) \leqslant c.$$

Such a proposition appears in [10]. We give here a similar result in the context of homogeneous systems. It is a consequence of the three following lemmas.

Let us introduce the function

$$\psi(u) = 2u^2 - 4u + 1, \qquad 0 \leqslant u \leqslant 1 - \frac{\sqrt{2}}{2}.$$

This function is decreasing from 1 at $u = 0$ to 0 at $u = 1 - \sqrt{2}/2$. We first start with a linear algebra lemma. Its proof may be found in [3, Lemma 2a].

LEMMA 1. *Let X and Y be Hermitian spaces and A, B: $X \to Y$ linear operators with B onto. If*

$$\|B^\dagger (B - A)\| \leqslant \lambda < 1$$

then A is onto and

$$\|A^\dagger B\| < \frac{1}{1-\lambda}.$$

LEMMA 2. *Let x and $y \in (\mathbb{C}^{n+1})^\star$ be given such that $Df(x)|_{x^\perp}$ is onto, $\|x\| = 1$, and $u = \|y - x\|\, \gamma(f, x) < 1 - \sqrt{2}/2$. Then $Df(y)|_{x^\perp}$ is onto and*

$$\|(Df(y)|_{x^\perp})^\dagger\, Df(x)|_{x^\perp}\| \leqslant \frac{(1-u)^2}{\psi(u)}.$$

Proof. $Df(y) = Df(x) + \sum_{k \geqslant 2} k(D^k f(x)/k!)(y - x)^{k-1}$ so that

$$(Df(x)|_{x^\perp})^\dagger\, (Df(y)|_{x^\perp} - Df(x)|_{x^\perp})\sum_{k \geqslant 2} k(Df(x)|_{x^\perp})^\dagger\, \frac{D^k f(x)}{k!}\, (y - x)^{k-1}\,|_{x^\perp}.$$

If we take the operator norm of both sides we get

$$\|(Df(x)|_{x^\perp})^\dagger\,(Df(y)|_{x^\perp} - Df(x)|_{x^\perp})\| \leqslant \sum_{k\geqslant 2} k\gamma(f,x)^{k-1}\,\|y-x\|^{k-1}$$

$$= \sum_{k\geqslant 2} ku^{k-1} = \frac{1}{(1-u)^2} - 1$$

and this number is <1 since $u<1-\sqrt{2}/2$. By Lemma 1, $Df(y)|_{x^\perp}$ is onto and

$$\|(Df(y)|_{x^\perp})^\dagger\,Df(x)|_{x^\perp}\| \leqslant \frac{1}{1-((1/(1-u)^2)-1)} = \frac{(1-u)^2}{\psi(u)}. \quad \blacksquare$$

LEMMA 3. *Let x and $\zeta \in (\mathbb{C}^{n+1})^\star$ be given such that $\|x\| = \|\zeta\| = 1$ and $\alpha(f,x) \leqslant \alpha_0$ with associated zero ζ. Then*

$$\gamma(f,\zeta) \leqslant \frac{\gamma(f,x)}{(1-\sigma\alpha_0)\,\psi(\sigma\alpha_0)},$$

where σ and α_0 are the constants appearing in Theorem 3.

Proof. Since $f(\zeta)=0$ we have $\mathbb{C}\zeta \subset \ker Df(\zeta)$ so that $(Df(\zeta)|_{\zeta^\perp})^\dagger = Df(\zeta)^\dagger$ is the minimum norm right inverse of $Df(\zeta)$. Thus,

$$\gamma(f,\zeta) = \max\left(1, \max_{k\geqslant 2}\left\|(Df(\zeta)|_{\zeta^\perp})^\dagger\,\frac{D^k f(\zeta)}{k!}\right\|^{1/(k-1)}\right)$$

$$\leqslant \max\left(1, \max_{k\geqslant 2}\left\|(Df(\zeta)|_{x^\perp})^\dagger\,\frac{D^k f(\zeta)}{k!}\right\|^{1/(k-1)}\right).$$

Since $\alpha(f,x) \leqslant \alpha_0$, $Df(x)|_{x^\perp}$ is onto so that

$$\left\|(Df(\zeta)|_{x^\perp})^\dagger\,\frac{D^k f(\zeta)}{k!}\right\| \leqslant \|(Df(\zeta)|_{x^\perp})^\dagger\,Df(x)\|\,\left\|(Df(x)|_{x^\perp})^\dagger\,\frac{D^k f(\zeta)}{k!}\right\|.$$

Let us denote $u = \|x-\zeta\|\,\gamma(f,x)$. Since $\alpha(f,x) \leqslant \alpha_0$, by Theorem 3, we have $d_R(\zeta, z) \leqslant \sigma\beta(f,x)$. Moreover, since $\|x\| = \|\zeta\| = 1$, we also have $\|x-\zeta\| \leqslant d_R(\zeta, z)$ so that

$$u \leqslant d_R(\zeta,x)\,\gamma(f,x) \leqslant \sigma\alpha(f,x) \leqslant \sigma\alpha_0 < 1 - \frac{\sqrt{2}}{2}.$$

Thus, by Lemma 2

$$\|(Df(\zeta)|_{x^\perp})^\dagger\, Df(x)\| < \frac{(1-u)^2}{\psi(u)}.$$

We also have

$$\begin{aligned}\left\|(Df(x)|_{x^\perp})^\dagger\, \frac{D^k f(\zeta)}{k!}\right\| &\leqslant \sum_{l\geqslant 0} \left\|(Df(x)|_{x^\perp})^\dagger\, \frac{D^{k+l} f(x)}{k!\, l!}\right\| \|x-\zeta\|^l \\ &\leqslant \sum_{l\geqslant 0} \left\|\frac{(k+l)!}{k!\, l!}\, \gamma(f,x)^{k+l-1}\right\| \|x-\zeta\|^l = \frac{\gamma(f,x)^{k-1}}{(1-u)^{k+1}}.\end{aligned}$$

Thus

$$\gamma(f,\zeta) \leqslant \max\left(1, \max_{k\geqslant 2}\left(\frac{(1-u)^2}{\psi(u)}\,\frac{\gamma(f,x)^{k-1}}{(1-u)^{k+1}}\right)^{1/(k-1)}\right)$$

and, using the inequality $u \leqslant \sigma\alpha_0$, we are done. ∎

Proof of Proposition 2. Since $\alpha(f,x)\leqslant\alpha_0$, by Theorem 3, we have $d_R(\zeta,x)\leqslant\sigma\beta(f,x)$. Using Lemma 3 we obtain

$$d_R(\zeta,x)\,\gamma(f,\zeta)\leqslant\sigma\beta(f,x)\,\frac{\gamma(f,x)}{(1-\sigma\alpha_0)\,\psi(\sigma\alpha_0)}\leqslant\frac{\sigma\alpha_0}{(1-\sigma\alpha_0)\,\psi(\sigma\alpha_0)}$$

and we are done. ∎

Our last ingredient is a corollary of the following theorem (gamma-theorem for homogeneous polynomial systems); see [3, Theorem 2] and [1, Chap. 14, Theorem 1] for the case $m=n$.

THEOREM 4. *There is a universal constant γ_0 with the following property: let $\zeta\in(\mathbb{C}^{n+1})^\star$ be a zero of $f\in(\mathscr{H}_d)^\star$ and $x\in(\mathbb{C}^{n+1})^\star$. If*

$$\|x-\zeta\|\,\gamma(f,\zeta)/\|\zeta\|\leqslant\gamma_0$$

then the projective Newton sequence,

$$x_0=x, \qquad x_{k+1}=N_f(x_k),$$

is defined and converges to a zero $\zeta'\in(\mathbb{C}^{n+1})^\star$ of f and

$$d_R(\zeta',x_k)\leqslant\sigma(\tfrac{1}{2})^{2^k-1}\,\beta(f,x).$$

PROPOSITION 3. *For any* $\zeta \in (\mathbb{C}^{n+1})^\star$ *and any* $f \in (\mathcal{H}_d)^\star$ *such that* $f(\zeta) = 0$ *we have*

$$d_R(\zeta, \Sigma_f)\, \gamma(f, \zeta) \geqslant \gamma_0.$$

Proof. If $x \in \Sigma_f$ then the projective Newton sequence $x_0 = x$, $x_{k+1} = N_f(x_k)$ is not defined. By Theorem 4 we have necessarily

$$\|x - \zeta\|\, \gamma(f, \zeta)/\|\zeta\| > \gamma_0.$$

When $\langle x, \zeta \rangle = 0$ then $d_R(\zeta, x) = \pi/2$ and the conclusion holds. When $\langle x, \zeta \rangle \neq 0$, scaling x such that $\langle x - \zeta, x \rangle = 0$, we obtain

$$d_R(x, \zeta) \geqslant \|x - \zeta\|/\|\zeta\|$$

and we are done. ▮

Proofs of Theorems 1 *and* 2. Given an NCM sequence (f_i, ζ_i), $0 \leqslant i \leqslant k$, since $\alpha(f_{i+1}, \zeta_i) \leqslant \alpha_0$ with associated zero ζ_{i+1}, we get by Proposition 2

$$d_R(\zeta_{i+1}, \zeta_i)\, \gamma(f_{i+1}, \zeta_{i+1}) \leqslant c.$$

By Proposition 1 we obtain

$$\max\left(1, \frac{D-1}{2}\right) d_R(\zeta_{i+1}, \zeta_i) \leqslant c,$$

so that

$$\max\left(1, \frac{D-1}{2}\right) d_R(\zeta_0, \zeta_k) \leqslant \max\left(1, \frac{D-1}{2}\right) \sum_{i=0}^{k-1} d_R(\zeta_{i+1}, \zeta_i) \leqslant ck$$

and this proves Theorem 1.

As previously we have

$$d_R(\zeta_{i+1}, \zeta_i)\, \gamma(f_{i+1}, \zeta_{i+1}) \leqslant c$$

and by Proposition 3 we have

$$d_R(\zeta_{i+1}, \zeta_i)\, \gamma_0\, d_R(\zeta_{i+1}, \Sigma_{f_{i+1}})^{-1} \leqslant d_R(\zeta_{i+1}, \zeta_i)\, \gamma(f_{i+1}, \zeta_{i+1}) \leqslant c,$$

so that

$$d_R(\zeta_0, \zeta_k) \leqslant \sum_{i=0}^{k-1} d_R(\zeta_{i+1}, \zeta_i) \leqslant c\gamma_0^{-1} k \left(\frac{1}{k} \sum_{i=0}^{k-1} d_R(\zeta_{i+1}, \Sigma_{f_{i+1}}) \right)$$

this proves Theorem 2. ∎

REFERENCES

1. Blum, L., Cucker, F., Shub, M., and Smale, S. (1997), "Complexity and Real Computation," Springer-Verlag, New York/Berlin.
2. Dedieu, J. P. (1997), Condition number analysis for sparse polynomial systems, *in* "Foundations of Computational Mathematics" (F. Cucker and M. Shub, Eds.), Springer-Verlag, New York.
3. Dedieu, J. P., and Shub, M. (1997), Multihomogeneous Newton's method, *Math. of Comp.*, to appear.
4. Dedieu, J. P., and Shub, M. (1998), Newton and predictor–corrector methods for overdetermined systems of equations, preprint.
5. Malajovich, G. (1994), On generalized Newton algorithms, *Theor. Comput. Sci.* **133**, 65–84.
6. Shub, M. (1993), Some remarks on Bézout's theorem and complexity, *in* "Proceedings of the Smalefest" (M. V. Hirsch, J. E. Marsden, and M. Shub, Eds.), pp. 443–455, Springer-Verlag, New York.
7. Shub, M., and Smale, S. (1993), Complexity of Béout's theorem. I. Geometric aspects, *J. Am. Math. Soc.* **6**, 459–501.
8. Shub, M., and Smale, S. (1996), Complexity of Bézout's theorem. IV. Probability of success, extensions, *SIAM J. Numer. Anal.* **33**, 128–148.
9. Shub, M., and Smale, S. (1994), Complexity of Bézout's theorem. V. Polynomial time, *Theor. Comput. Sci.* **133**, 141–164.
10. Smale, S. (1986), Newton's method estimates from data at one point, *in* "The Merging of Disciplines: New Directions in Pure, Applied and Computational Mathematics" (R. Ewing, K. Gross, and C. Martin, Eds.), Springer-Verlag, New York/Berlin.